Chapter 17

Area Under a Curve[1]

In Chapters 14 and 15 we learned how to find the rate of change function, the derivative, of various functions. We used the derivative in business applications to find such things as marginal cost or marginal profit functions and to study optimization of cost or revenue. In this chapter we do the opposite: For example, instead of beginning with the cost function c and then finding the marginal cost function by applying the rules of differentiation to c, we begin with and find c by reversing the rules of differentiation. This process is called antidifferentiation or finding the indefinite integral. Mathematics and science majors usually take an entire course in differentiation and then follow it with another course devoted to integration. In these courses, students study a rather amazing idea, called the Fundamental Theorem of Calculus; namely, an anitderivative c of a derivative function (found by reversing the rules of differentiation) is intimately connected to the area under the graph of the derivative function $f'(x)$. Although we will study some of the basic rules of antidifferentiation in order to find the area under various curves by using the Fundamental Theorem of Calculus, we will also use spreadsheets to find approximate numerical answers to finding the area, approximations that serve us quite well in real-life situations. This process is called numerical integration. Indeed, for some important functions there is no known way of finding their antiderivative and, as a result, numerical integration is the only way we have of finding the area under the graph of these particular functions.

1. In the first section of the chapter, we will use both numerical integration and the Fundamental Theorem of Calculus to find the area under a curve.

2. In the second section, we apply finding the area under a curve to some business applications.

- *As a result of this chapter, students will learn*
  - √ How to find antiderivatives, i.e. the indefinite integral, of certain basic functions
  - √ How to use the Fundamental Theorem of Calculus to compute the area under a curve, i.e. the definite integral
  - √ How to use numerical integration to compute the area under a curve

- *As a result of this chapter, students will be able to*
  - √ Use an integration tool to find the definite integral
  - √ Compute the area between two curves
  - √ Apply numerical integration to find the total cost of production
  - √ Compute future and present value of an income stream
  - √ Compute consumers' and producers' surplus

17.1 Calculating the Area under a Curve

In this section we investigate three interrelated concepts concerning the area under the graph of a function:

1. Approximating the area under the curve by a finite number of rectangles (Example 1 (page 502) and Exploration 17A) and then seeing how an infinite number of these rectangles could give the exact area under the curve

2. Finding the antiderivative of the function (Example 2 (page 503))

3. The Fundamental Theorem of Calculus, which connects 1 and 2 above (Example 3 (page 503))

In (1) above, we approximate the area under a curve by summing the areas of certain rectangles. The more of these rectangles we use, the more closely their sum approximates the true area under the curve. If we sum an infinite number of these rectangles, we will have the exact area under the curve. In (2) above, we find the antiderivative function f, also called the indefinite integral, denoted by $\int f'(x)dx$, by reversing the rules on the derivative function f'. In (3) above, the Fundamental Theorem of Calculus ties the area under the curve of f' to its antiderivative f as follows:

$$\text{The area under} f' \text{from} a \text{to} b = f(b) - f(a).$$

That is, the area under the curve $y = f'(x)$ from $x = a$ to $x = b$, $\int_a^b f'(x)dx$, is the difference of the antiderivative function, i.e. the indefinite integral, evaluated at b and at a. Symbolically, the Fundamental Theorem is written: $\int_a^b f'(x)dx = f(b) - f(a)$, where a is called the lower limit of the integral and b is called the upper limit. Another common way of symbolizing the Fundamental Theorem is: $\int_a^b f(x)dx = F(b) - F(a)$ where $F' = f$. Here F is the antiderivative function of the derivative function f.

We turn now to the procedure for finding the approximate area under a curve. We will illustrate the procedure by taking as our derivative function, f', a marginal cost function that we will denote by $c'(x)$, where $c'(x)$ is the marginal cost of producing x items of a commodity. We wish to find the area under $y = c'(x)$ from a to b (see the left half of figure 17.3). This area will turn out to be the total variable cost of producing from a to b items of the commodity.

We divide the interval from a to b into n equal subintervals of width (see the right half of figure **??** in which we use 5 subintervals) where $\Delta x = \frac{b-a}{n}$.

For ease of notation, we will rewrite these subintervals as follows: $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, $x_3 = a + 3\Delta x$, $x_4 = a + 4\Delta x$. If we draw a vertical line segment from $x = a$ to the segment's intersection with the curve (see figure 17.2), the height of this segment represents the marginal cost at a, i.e. the cost of producing one more item when we have already produced a items.

We then create a rectangle with width Δx, the additional items produced beyond a. The area of this rectangle is $c'(x)\Delta x$, which is the approximate cost of producing the Δx items from x_0 to x_1. (See the right half of figure 17.2.)

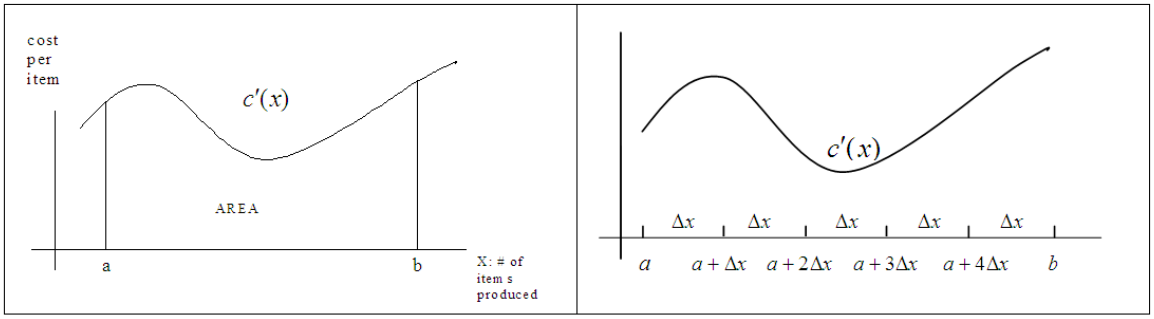

Figure 17.1: Graph showing a marginal cost function between two limits, a and b (left) and showing the interval from a to b broken into 5 subintervals of equal size Δx.

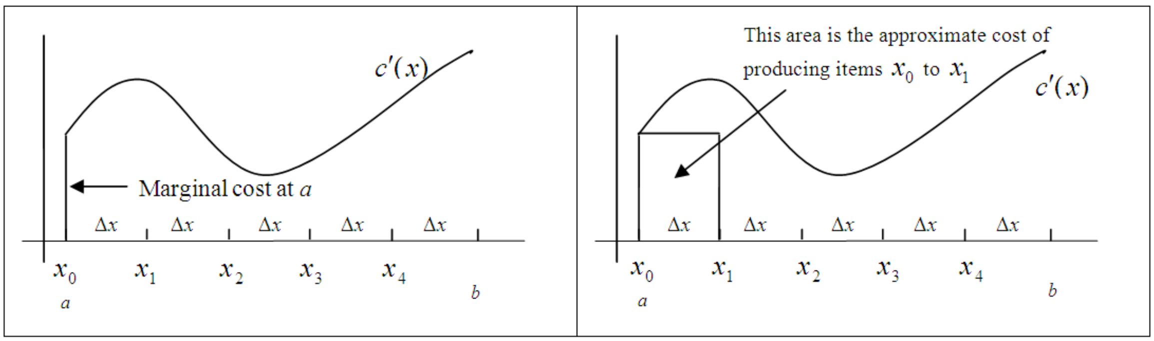

Figure 17.2: Illustration of breaking the area under the margnial cost function into subrectangles that approximate the area.

We will deal with the problem of the area of the rectangle underestimating the true area under the curve from x_0 to x_1 shortly. Nevertheless, we continue by constructing a second rectangle with a vertical line segment through x_1 to the curve c' with width Δx. The area of this rectangle is the approximate cost of producing items x_1 to x_2, which happens to overestimate the true area under the curve from x_1 to x_2 (see Figure 17.5).

Filling in the remaining rectangles in a similar fashion, we find that the sum of these five rectangles is the approximate area under the curve from a to b (see Figure 17.6).

The approximate area under curve from a to b is

$$c'(x_0)\Delta x + c'(x_1)\Delta x + c'(x_2)\Delta x + c'(x_3)\Delta x + c'(x_4)\Delta x.$$

We can imagine constructing an arbitrary number of rectangles n as we have the five above, each of which has width $\Delta x = \frac{b-a}{n}$:

$$\sum_{i=0}^{n-1} c'(x_i)\Delta x = c'(x_0)\Delta x + c'(x_1)\Delta x + c'(x_2)\Delta x + c'(x_3)\Delta x + c'(x_4)$$

This is an example of a Riemann Sum ("Rie" rhymes with "me" and "mann" rhymes with "Don." There are many different ways of constructing rectangles under the curve from

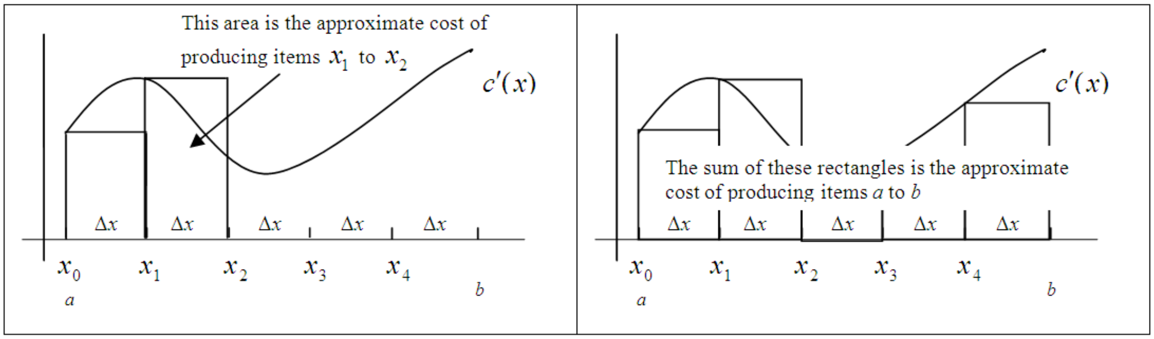

Figure 17.3: Graph showing a marginal cost function between two limits, a and b (left) and showing the interval from a to b broken into 5 subintervals of equal size Δx.

a to b, some more accurate and efficient than others. Nonetheless, it turns out that each will lead to the same place, the true area under the curve from a to b. This is how:

We increase the number n of rectangles and correspondingly shrink the width Δx. The over and under estimations of the rectangles decrease as the number of rectangles increases. Mathematically, though not geometrically, it turns out that an infinite number of "rectangles" can be packed under the curve and summed to give the exact area under the curve from a to b. This is denoted by $\lim_{n\to\infty}\sum_{i=0}^{n-1} c'(x_i)\Delta x = \int_a^b c'(x)dx$ and is read "the limit of the Riemann sum $\sum_{i=0}^{n-1} c'(x_i)\Delta x$ as n goes to infinity (∞ is the symbol for infinity) is the definite integral of $c'(x)$ from a to b."

17.1.1 Definitions and Formulas

Indefinite Integral or antiderivative F is an antiderivative function of f if $F'(x) = f(x)$. The antiderivative is also called the indefinite integral of f and is denoted by $\int f(x)dx + C$, where C is a constant.

Constant of integration If $F(x)$ is an anitderivative function of f, then $F(x)+C$, where C is any real number, is likewise an anitderivative function of f since $\frac{d}{dx}(F(x)+C) = \frac{d}{dx}F(x) + 0 = f(x)$. C is called the constant of integration for the indefinite integral.

Definite integral and the limit of a Riemann sum The definite integral from $x = a$ to $x = b$ is the limit of the Riemann sum $\lim_{n\to\infty}\sum_{i=0}^{n-1} c'(x_i)\Delta x$ as $n \to \infty$, where $\Delta x = \frac{b-a}{n}$. The definite integral is denoted by $\int_a^b f(x)dx$.

Lower limit and upper limit a is called the lower limit of the integral $\int_a^b f(x)dx$ and b is called the upper limit.

Fundamental Theorem of Calculus The Fundamental Theorem of Calculus states that $\int_a^b f(x)dx = F(b) - F(a)$, where F is an antiderivative of f.

Area Under a Curve If $f(x)$ is positive from a to b, then the definite integral $\int_a^b f(x)dx$ computes the area under $f(x)$ and above the x-axis from a to b.

Numerical Integration Numerical integration approximates the definite integral

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x$$

by $\sum_{i=0}^{N-1} f(x_i)\Delta x$, where N is a very large number. There are several different methods of numerical integration but this text uses the method of rectangles for simplicity and ease of discussion.

17.1.2 Worked Examples

Example 17.1. Approximating the Area Under a Curve

Let $c'(x) = \frac{1}{2}x + 1$. Approximate the area under the graph of $y = c'(x)$ from $x = 1$ to 3 with four rectangles.

We first find the width of each rectangle: $\Delta x = \frac{3-1}{4} = 0.5$.

We next calculate the Riemann sum from 1 to 3:

$$\begin{aligned} \text{Area} \approx\ & c'(1)(0.5) + c'(1.5)(0.5) + c'(2)(0.5) + c'(2.5)(0.5) \\ &= \left[\frac{1}{2}(1) + 1\right](0.5) + \left[\frac{1}{2}(1.5) + 1\right](0.5) + \left[\frac{1}{2}(2) + 1\right](0.5) + \left[\frac{1}{2}(2.5) + 1\right](0.5) \\ &= [0.5 + 1](0.5) + [0.75 + 1](0.5) + [1 + 1](0.5) + [1.25 + 1](0.5) \\ &= 3.75. \end{aligned}$$

So, the approximate area from 1 to 3 found by summing the areas of the four rectangles is 3.75. If we were to increase the number of rectangles to 16, then 64, then 1000, and then 10000, we would obtain the following results for the approximate area under the curve from 1 to 3:

Number of Rectangles	Width of Each Rectangle	Approximation to Area
$n = 4$	.5	3.75
$n = 16$	.125	3.9375
$n = 64$	.03125	3.984375
$n = 1000$	.002	3.999
$n = 10000$	.0002	3.9999

According to the Fundamental Theorem of Calculus (to be illustrated in example 3 (page 503)), the exact area under the curve from 1 to 3 is 4, which you might believe from the table, which suggests that as the number of rectangles used to estimate the area increases, the estimated area approaches the number 4. By using a very large number of rectangles, we can approximate the area under a curve quite closely. Although the calculations are staggering when computed with pencil and paper, computers can handle the computations in a fraction of a second (see Exploration 17A). The process we have described is an example of numerical integration. There are several methods other than using rectangles to compute definite integrals by numerical integration.

Example 17.2. Finding Antiderivatives for Some Basic Functions

If F is an antiderivative function of f, then $F'(x) = f(x)$. But then $F(x) + 2$ is also an antiderivative function of f since the derivative of a constant, like the number 2, is zero. In general, if $F(x)$ is an antiderivative of f, then so is $F(x) + C$, where C is any real number. C is called the constant of integration for F. Note: We will write the constant of integration in upper case in order to distinguish it from a function c or $c(x)$, which we will write in lower case.

General Rule	Examples
$\int a\,dx = ax + C$, where a is any constant	$\int x\,dx = x + C$ since $\int dx = \int 1\,dx$ $\int -2\,dx = -2x + C$ $\int 5\,dx = 5x + C$
$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$ where $n \neq -1$	$\int x\,dx = \frac{1}{2}x^2 + C$ since $x = x^1$ $\int x^2\,dx = \frac{1}{3}x^3 + C$ $\int x^{-3}\,dx = \frac{1}{-2}x^{-2} + C = -\frac{1}{2x^2} + C$ since $-3 + 1 = 2$
$\int x^{-1}\,dx = \ln\|x\| + C$	$\int \frac{dx}{x} = \ln\|x\| + C$ since $\frac{1}{x} = x^{-1}$ See note below[2]
$\int af(x)\,dx = a\int f(x)\,dx + C$ where a is any constant	$\int 2x^3\,dx = 2\int x^3\,dx = 2\left(\frac{x^4}{4}\right) + C = \frac{3}{2}x^4 + C$ $\int \frac{2}{x}\,dx = 2\int \frac{dx}{x} = 2\ln x + C$
$\int e^{bx}\,dx = \frac{1}{b}e^{bx} + C$	$\int e^{0.2x}\,dx = \frac{1}{0.2}e^{0.2x} + C = 5e^{0.2x} + C$ $\int 10e^{-0.005x}\,dx = 10\int e^{-0.005x}\,dx =$ $= \frac{10}{-0.005}e^{-0.005x} + C = -2000e^{-0.005x} + C$
$\int (f(x) + g(x))\,dx$ $= \int f(x)\,dx + \int g(x)\,dx$	$\int \left(-4x^{-2} + 2x^{-1}\right)\,dx = \frac{-4}{-1}x^{-1} + 2\ln x + C = \frac{4}{x} + \ln x + C$ $\int(-x^2 + 4x - 2)\,dx = \frac{-1}{3}x^3 + \frac{4}{2}x^2 - 2x + C$

Example 17.3. Using the Fundamental Theorem of Calculus to Find Total and Variable Costs from Marginal Cost

Suppose we have gathered daily marginal cost for manufacturing a particular item and found its regression model to be $c'(x) = 0.0002x^2 - 0.1x + 30$, where $c'(x)$ is measured in dollars per item and x is the number of units produced. The fixed cost of producing any number of items is \$550.

Part a. Find the total cost in producing the first 350 units per day.

Before using the Fundamental Theorem of Calculus, we need to find the antiderivative of $c'(x) = 0.0002x^2 - 0.1x + 30$.

$$\int (0.0002x^2 - 0.1x + 30)dx = \frac{0.0002}{3}x^3 - \frac{0.1}{2}x^2 + 30x + C$$

Since the fixed cost of producing zero items is \$550, we know that $C = 550$. (Substitute $x = 0$ into the antiderivative and set it equal to 550; this gives $C = 550$). Applying the Fundamental Theorem of Calculus, we find the variable cost of producing the first 350 items:

$$\begin{aligned}
\int_0^{350}(0.0002x^2-0.1x+30)dx &= \left.\frac{0.0002}{3}x^3-\frac{0.1}{2}x^2+30x+C\right]_0^{350}\\
&= \left[\frac{0.0002}{3}(350)^3-\frac{0.1}{2}(350)^2+30(350)+550\right]-[0+550]\\
&= \frac{0.0002}{3}(350)^3-\frac{0.1}{2}(350)^2+30(350)+550-550\\
&= \$7233.\bar{3}\bar{3} \approx \$7233
\end{aligned}$$

NOTE: In the definite integral, the constants of integration always cancel each other out (see $550-550$ above). Therefore, when we compute the definite integral, we will omit the C, the constant of integration.

$7233 is the variable cost of producing the first 350 items. The total cost must include the fixed cost, $550 of producing any number of items:

$$\text{Total Cost} = \$7233 + \$550 = \$7783.$$

Part b. What is the variable cost of producing the 151st through the 350th unit?

$$\begin{aligned}
\int_{150}^{350}(0.0002x^2-0.1x+30)dx &= \left.\frac{0.0002}{3}x^3-\frac{0.1}{2}x^2+30x+C\right]_{150}^{350}\\
&= \left[\frac{0.0002}{3}(350)^3-\frac{0.1}{2}(350)^2+30(350)\right]\\
&\quad -\left[\frac{0.0002}{3}(150)^3-\frac{0.1}{2}(150)^2+30(150)\right]\\
&= \$3633
\end{aligned}$$

NOTE: Since the cost of producing the 151st item begins just after having produced the 150th item, the left-hand height of the first rectangle in the Riemann sum begins at $x = 150$.

17.1.3 Exploration 17A: Numerical Integration

Find $\int_0^{350}(0.0002x^2 - 0.1x + 30)dx$ by numerical integration.

Bring up the file C17 Integration Tool.xls. Copy it to a new worksheet and save it under other file name. Follow the How To Guide in order to see how to modify this worksheet to perform numerical integration for this function.

Your answer should closely match the answer to this same integral in example 1 (page 502) (part a), which is the exact area as found by the Fundamental Theorem of Calculus.

17.1.4 How To Guide

A Basic Integration Tool in Excel

Bring up the file C17 Integration Tool.xls. This file contains the numerical integration of the definite integral $\int_0^1 x^2\,dx$; that is, it computes a reasonable numerical approximation to the exact area under the curve $f(x) = x^2$.

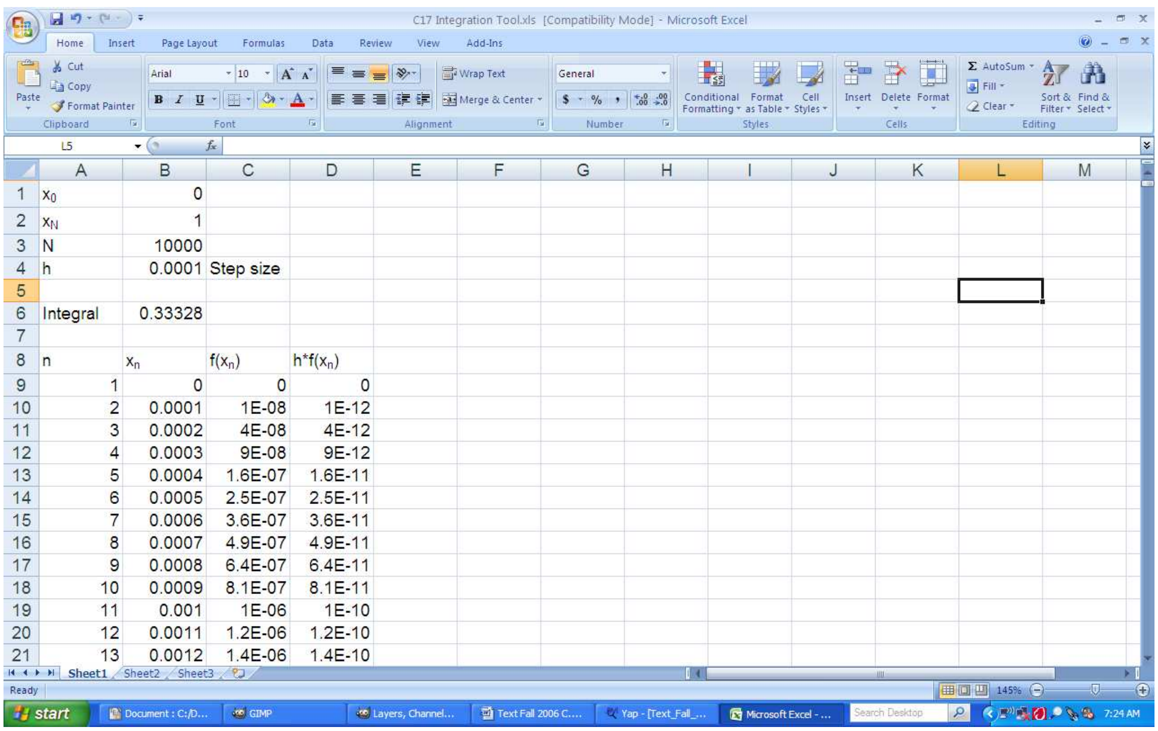

Figure 17.4: Excel tool for numerically approximating an integral.

Below, we will learn how to modify this spreadsheet in order to integrate other functions with other lower and upper limits.

Cell References and Formulas in the Basic Tool

A1 : is the lower limit 0, which is entered numerically in B1
A2 : is the upper limit 1, which is entered numerically in B2
A3 : N is the number of rectangles we will use to approximate the area under the curve from 0 to 1, which is entered numerically as 10000 in B3. Use this figure for all functions you wish to integrate numerically.
A4 : h is the width Δx of a rectangle, which is entered numerically in B4 as =(B2-B1)/B3. You do not need to change this.
A6 : The numerical value of the area under the curve from a to b, which is entered in B6 as =SUM(D9:D10008)
A8 : n counts the number of rectangles from 1 to 10000 in a column. This column should remain unchanged when you compute other functions.
B8 : indicates the left-hand endpoint of the base of the nth rectangle
C8 : $f(x_n)$ evaluates the function at the left-hand endpoint x_n to give the height of the nth rectangle at x_n, which is entered in C9 as =B9*B9. This formula will change for different functions that you are integrating.
D8 : $h \cdot f(x_n)$ multiplies the width h (i.e. Δx) and the height $f(x_n)$ of the nth rectangle to give the area of the nth rectangle, which is entered in D9 as =B4*C9

How to Modify the Basic Integration Tool for Other Functions

Example: Find $\int_2^4 (x^2+1)\,dx$ by numerical integration by modifying the Basic Integration Tool.

1. Copy C17 Integration Tool.xls to a new worksheet and save under other file name.
2. Set B1 to 2
3. Set B2 to 4
4. Set C9 = B9*B9+1 This is the new function we wish to integrate.
5. Highlight C9 and place the cursor on the lower right corner (fill handle) until it becomes +. DOUBLE CLICK the plus. All calculations will be computed at this point.
6. The approximate area under the curve is 20.66547, which is found in B6. The exact area under the curve as found by the Fundamental Theorem of Calculus is $20\frac{2}{3} \approx 20.67$.

17.2 Applications of the Definite Integral

In this section we will apply the definite integral to three useful analytical tools for business:

1. the future value of an income stream,
2. the present value of an income stream, and
3. consumers' and producers' surplus.

The first of these, the future value of an income stream, measures the value of an income stream by calculating the accumulated total of a continuing stream of revenue invested at a continuous rate r over a period of T years. The present value of an income stream is another way of measuring the value of an income stream. The present value is the lump-sum principle that would have to be invested now for a period of T years at a continuous rate of interest r in order to equal the future value of the income stream continually invested at the same rate over the same time T. That is, the bigger the future value of the stream, the more up-front principal P would have to be invested now at the same rate r in order to come out the same as the future value after T years.

Consumers' and suppliers' surplus help management evaluate the unit price of a commodity, i.e. whether it is too low or too high or just about right for the market.

17.2.1 Definitions and Formulas

Demand Function Expresses consumer demand for a product in terms of its unit price p and the number of units x that consumers will buy at price p. The demand function $D(x)$ creates a deceasing (downhill) curve because fewer products will be sold if p is larger; conversely, smaller prices will create more demand.

Consumers' Surplus Let $\bar{p}$ be a fixed established price for a commodity and $\bar{x}$ be the number of units bought at $\bar{p}$. The consumers' surplus is the difference between what consumers would be willing to pay for a commodity and what they actually pay for it. The formula for consumers' surplus is: $CS = \int_0^{\bar{x}} D(x)\,dx - \bar{p}\bar{x}$, where $D(x)$ is the demand function.

Supply Function Expresses producers willingness to supply x units of a commodity at price p. The supply function $S(x)$ creates an increasing (uphill) curve because producers are willing to put more of the commodity on the market if the unit price is higher.

Producer's Surplus Let $\bar{p}$ be a fixed established price for a commodity and $\bar{x}$ be the number of units that producers are willing to supply at price $\bar{p}$. The producers' surplus is the difference between what the suppliers actually receive $\bar{x}$ and what they would be willing to receive at price $\bar{p}$. The formula for producers' surplus is: $PS = \bar{p}\bar{x} - \int_0^{\bar{x}} S(x)\,dx$ where $S(x)$ is the supply function.

Market Equilibrium, equilibrium quantity, equilibrium price The point $(\bar{x}, \bar{p})$ where the demand curve and the supply curve intersect, i.e. the point at which market equilibrium occurs. This is the highest price consumers are willing to pay for what producers are willing to supply. $\bar{x}$ is called the equilibrium quantity and $\bar{p}$ is called the equilibrium price.

Income Stream Created when a business generates a stream of income $R(t)$, where t is in years, over a period of time T years and this income is invested at an annual rate r compounded continuously. $R(t)$ could be a constant stream or a variable stream but it is invested, nonetheless, on a continuing basis over the T years.

Future Value of an Income Stream The total amount of money that will be accumulated when an income stream $R(t)$ has been invested at an annual rate r compounded continuously for T years. The formula for the future value of an income stream $R(t)$ is: $FV = e^{rt} \int_0^T R(t)e^{-rt}\,dt$.

Present Value of an Income Stream The principal P that would have to be invested at an annual rate r compounded continuously over T years in order to equal the accumulated value of an income stream over the same period T and the same rate r. The formula for the future value of an income stream $R(t)$ is: $PV = \int_0^T R(t)e^{-rt}\,dt$.

17.2.2 Worked Examples

Example 17.4. Future Value of an Income Stream

Let

$$\begin{aligned} R(x) &= \text{Rate of income at time t} \\ r &= \text{Interest rate compounded continuously} \\ T &= \text{Number of years the income stream is invested} \end{aligned}$$

We divide the interval $[0, T]$ into n subintervals of length $\Delta t = \frac{T}{n}$ and create n rectangles under the $R(x)$ as seen in the figure below.

The height of the ith rectangle is $R(t_i)$ and its width is Δt. $R(t_i)\Delta t$, the area of the ith rectangle, is the approximately the amount of the income stream to be invested between t_{i-1} and t_i. If we think of investing this small principal at the continuous rate r for a period of $T - t_i$ years (this is the remaining time for the investment from t_i to T), then the amount that will be accumulated after T years is $[R(t_i)\Delta t]e^{r(T-t_i)}$. This formula is derived from the compound interest formula $A = Pe^{rt}$ (see example 4 (page 455)) where $P = R(t_i)\Delta t$ and e^{rt} is replaced by $e^{r(T-t_i)}$. Therefore, the Riemann sum of the future values of the areas of the rectangles is

$$\begin{aligned} [R(t_1)\Delta t]e^{r(T-t_1)} &+ [R(t_2)\Delta t]e^{r(T-t_2)} + [R(t_3)\Delta t]e^{r(T-t_3)} + \ldots + [R(t_n)\Delta t]e^{r(T-t_n)} \\ &= [R(t_1)\Delta t]e^{rT}e^{-t_1} + [R(t_2)\Delta t]e^{rT}e^{-t_2} + [R(t_3)\Delta t]e^{rT}e^{-t_3} + \ldots + [R(t_n)\Delta t]e^{rT}e^{-t_n} \end{aligned}$$

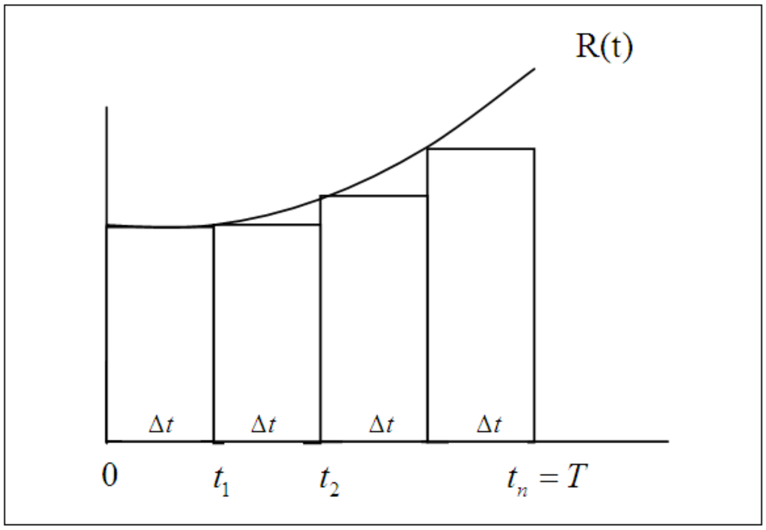

Figure 17.5: Using integration to find the future value of an income stream.

Factoring out e^{rT} and Δt, we have

$$= e^{rT}\left[[R(t_1)]e^{-t_1} + [R(t_2)]e^{-t_2} + [R(t_3)]e^{-t_3} + \ldots + [R(t_n)]e^{-t_n}\right]\Delta t$$

Letting $n \to \infty$, we have the following result:

The future value after T years of an income stream $R(t)$ dollars per year, invested at the rate r per year compounded continuously, is

$$FV = e^{rT}\int_0^T R(t)e^{-rt}\,dt$$

To illustrate this idea, consider Glow Health Spa. Glow Health Spa recently bought a full-body dermal treatment machine that is expected to generate \$50,000 in revenue per year for the next 5 years. If this income stream is invested at 8% per year compounded continuously, what is the future value of this income stream in 5 years?

$R(t)$	= 50,000 where t is in years Note: the income stream $R(t)$ is a constant amount per year in this example.
r	= .08
T	= 5 years
Future Value	$= e^{0.08(5)}\int_0^5 50000e^{-0.08t}\,dt = 1.49\int_0^5 50000e^{-0.08t}\,dt$ = 1.49(206054)=\$307,020 where the definite integral is calculated using the Basic Integration Tool.

Example 17.5. Present Value of an Income Stream
The present value of an income stream of $R(t)$ dollars per year over T years, earning interest at the rate of r per year compounded continuously, is the principal P that would have to be invested now to yield the same accumulated value as the investment stream would earn if it were invested on a continuing basis for T years at rate r.

In equation form, we have

$$Pe^{rT} = e^{rT} \int_0^T R(t)e^{-rt}\, dt.$$

Dividing both sides of this equation by e^{rT}, we have $PV = \int_0^T R(t)e^{-rt}\, dt$, the present value of the income stream $R(t)$.

To illustrate the present value of an income stream, we will consider the present value of Health Glow's income stream (see above).

$$PV = \int_0^5 50000e^{-0.08t}\, dt = \$206054$$

Where we have used the Basic Integration Tool in the last step. This means that in order to equal the future value (\$307,020) of investing an income stream of \$50000 for a period of 5 years at 8% compounded continuously, Health Glow would have to invest a lump sum now of \$206054 for the same time period at the same rate of interest.

Health Glow's revenue stream was constant (\$50000) over the five-year period. Revenue streams need not be constant, however. Suppose Health Glow generated an increasing income stream given by $R(t) = 50,000 + 2000t$. We find the future value and present value of this income stream as follows:

$$\begin{aligned} FV &= e^{0.08(5)} \int_0^5 (50000 + 2000t)e^{-0.08t}\, dt = (1.49)(225287) = \$335,678 \\ PV &= \int_0^5 (50000 + 2000t)e^{-0.08t}\, dt = \$225,287 \end{aligned}$$

Example 17.6. Consumer Surplus
Let

$p = D(x)$	be the demand function for a commodity
$\bar{p}$	the established market price of the commodity
$\bar{x}$	the number of items sold at (i.e. the consumer demand at $\bar{p}$)

Consumers' surplus is the difference between what consumers would be willing to pay p and the actual price $\bar{p}$ they pay. If we plot $p = D(x)$ and the straight line $p = \bar{p}$ on the same axes, then the consumers' surplus is the area between $D(x)$ and $p = \bar{p}$, i.e. $D(x) - \bar{p}$, from 0 to $\bar{x}$.

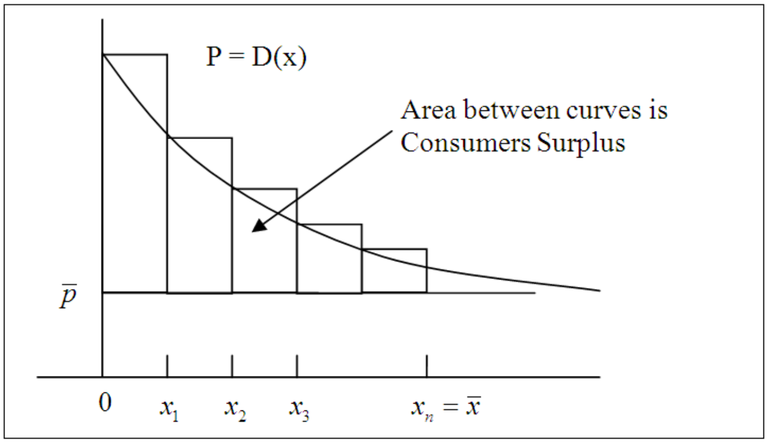

Figure 17.6: Integrating to compute the area between the demand curve $p = D(x)$ and the curve $p = \bar{p}$, the established market price of a commodity, determines the consumer surplus.

We find the approximate area between the two curves (the straight line $p = \bar{p}$ is considered to be a curve) by placing rectangles between them (see the figure below). The height of each rectangle is $D(x) - \bar{p}$ and the width is Δx. The area of the ith rectangle is $[D(x) - \bar{p}]\Delta x$.

The approximate area between the curves is the Riemann sum

$$\sum_{i=1}^{n}[D(x_i) - \bar{p}]\Delta x = [D(x_1) + D(x_2) + \ldots + D(x_n)]\Delta x - \underbrace{[\bar{p}\Delta x + \bar{p}\Delta x + \ldots + \bar{p}\Delta x]}_{\substack{n \text{ terms add to} \\ \bar{p}(n\Delta x) = \bar{p}\bar{x} \\ \text{because } n\Delta x = \bar{x}}}$$

Taking the sum as $n \to \infty$, we have

$$\lim_{n\to\infty}\sum_{i=1}^{n}[D(x_i) - \bar{p}]\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} D(x_i)\Delta x = \int_0^{\bar{x}} D(x)\,dx - \bar{p}\bar{x}$$

So, $CS = \int_0^{\bar{x}} D(x)\,dx - \bar{p}\bar{x}$. To illustrate this concept, suppose the demand function is given by $P = D(x) = -0.01x^2 - 0.3x + 25$ where

p	is the wholesale price in dollars
x	is the demand in thousands
$\bar{p} = 10$	the established market price (in dollars)

We need to find the intersection of $p = D(x)$ and $p = 10$ in order to find the area between the two curves. See the figure below.

Using Excel's Goal Seek (see Chapter 17 B How To Guide), we find the demand at \$10 is $\bar{x} \approx 26$ (rounded down to the nearest whole number). Substituting in the formula above and using the Basic Integration Tool for the definite integral, we have

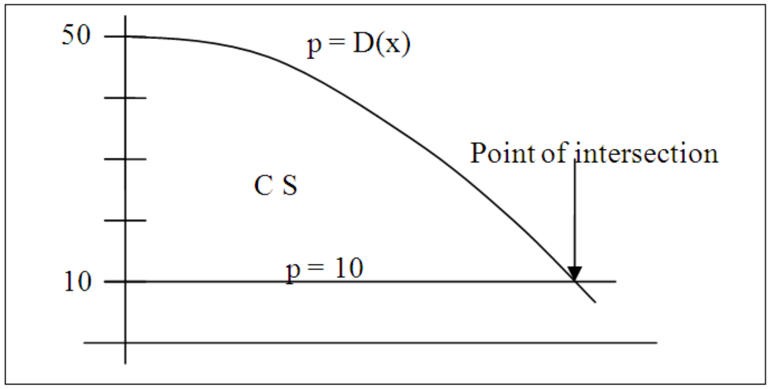

Figure 17.7: Finding the point of intersection to determine the limits of integration for computing consumer surplus.

$$\int_0^{26} (-0.01x^2 - 0.3x + 25)\, dx - (10)(26) = 500 - 260 = \$240 \text{ thousand}$$

Example 17.7. Producers' surplus
Let

$p = S(x)$	be the supply function for a commodity
$\bar{p}$	be the established market price of the commodity
$\bar{x}$	be the number of items producers are will to supply at $\bar{p}$ (i.e. the consumer demand at $\bar{p}$)

The producers' supply is the difference between what the suppliers actually receive and what they are willing to receive. The producers' surplus is the area between the line $p = \bar{p}$ and the supply curve $S(x)$.

Similar to what we did for the consumers' surplus, we find the area between the two curves to be the definite integral

$$\text{Producers' Surplus} = \bar{p}\bar{x} - \int_0^{\bar{x}} S(x)\, dx$$

To illustrate this concept, suppose the supply function is given by $P = S(x) = 0.1x^2 + 3x + 15$ where

p	is the price in dollars
x	is the demand in thousands
$\bar{p} = 50$	is the established market price

Before we can find the Producers' Surplus $= \bar{p}\bar{x} - \int_0^{\bar{x}} S(x)\, dx$, we need to find the intersection of $p = 50$ and $p = S(x)$ in order to find the area between the two curves. Using

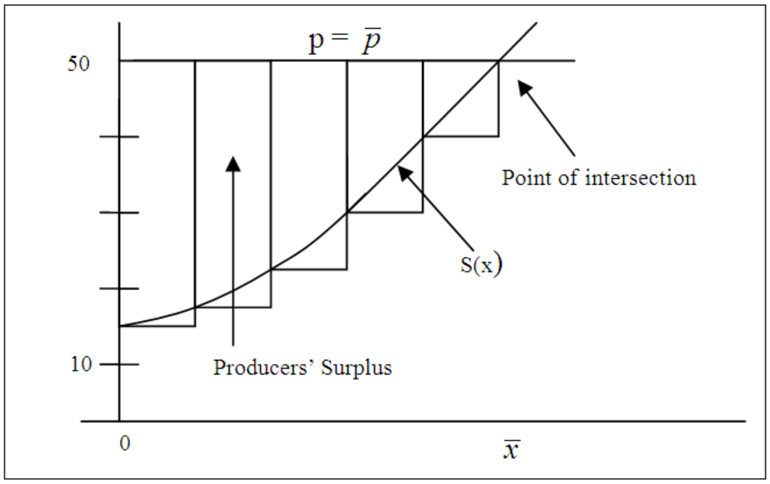

Figure 17.8: Using integration to find producer surplus, the area between the fair maarket price curve $p = \bar{p}$ and the supply curve $p = S(x)$.

Excel's Goal Seek (see Chapter 17 B How To Guide), we find the demand at \$50 is $\bar{x} \approx 9$. Substituting into the formula above and using the Basic Integration Tool for the definite integral, we have

$$(50)(9) - \int_0^9 (0.1x^2 + 3x + 15)\, dx = 450 - 281 = \$169 \text{ thousand}$$

17.2.3 Exploration 17B: Consumers' and Producers' Surplus at Market Equilibrium

1. Bring up the file Exploration 17B.xls. This file contains data from consumer and producer market surveys of a particular company's products. Find the equations for the demand function and the supply functions.

   $D(x) =$

   $S(x) =$

2. How do you find the point where market equilibrium occurs? (Hint: Use the difference of the demand and supply functions in Goal Seek as described in the How To Guide.)

   $\bar{x} =$

   $\bar{p} =$

3. Compute the consumers' and suppliers' surplus at market equilibrium $(\bar{x}, \bar{p})$.

4. Graph CS, PS, and $p = \bar{p}$ on the same axes where $\bar{p}$ is the equilibrium price (see the How To Guide).

5. Suppose the figure below illustrates the consumers' and producers' surplus at market equilibrium of a different commodity than above. Sketch the horizontal line $p = p_L$, where p_L is the established price of a commodity that is lower than the equilibrium price $\bar{p}$. What are the implications for the company in this situation?

6. The figure below illustrates the consumers' and producers' surplus at market equilibrium. Sketch the horizontal line $p = p_H$, where p_H is the established price of a commodity that is higher than the equilibrium price $\bar{p}$. What are the implications for the company in this situation?

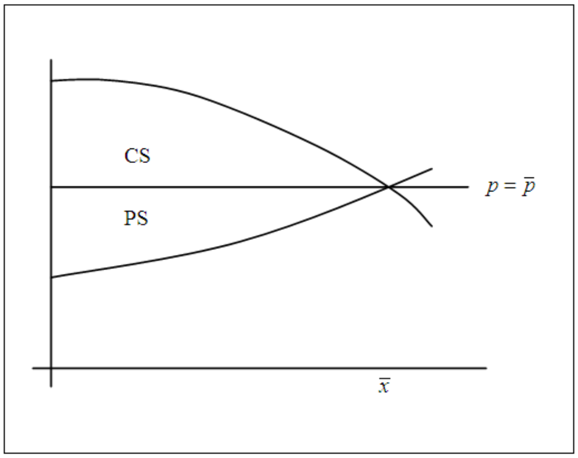

Figure 17.9: Graph for exploring what happens if the established price is lower than market equilibrium.

17.2.4 How To Guide

Finding the Intersection Point of Two Curves

Suppose you want to find the point of intersection of the two curves above given by the equations:

$$\begin{aligned} D(x) &= -0.0029x^2 - 0.0139x + 118.26 \\ S(x) &= 0.0015x^2 + 0.0806x + 30.596 \end{aligned}$$

The point of intersection occurs when the difference of the two functions, $D(x) - S(x)$, is zero. We will use Goal Seek to find the x that will make this difference zero. If you set up your spreadsheet as below, you will also find the y that goes with this x. To use Goal Seek, see the How to Guide for Chapter 7.B.

In A1, we have a "guess" value that Goal Seek needs to get going. We entered 120 because $x = 120$ is pretty close to the intersection of the two curves representing the graphs of our two functions. Cell C2 contains the formula for $D(x)$, cell D2 contains the formula for $S(x)$, and cell B2 contains $D(x) - S(x)$. B2 is the cell we want to set to zero by changing B1. You should get the following screens:

So the x-coordinate of the point of intersection is $x = 131$ in cell A2. The y-coordinate of the point of intersection is $y = 67$, as we can see in cells C2 and D2. In our example, the point of equilibrium is $(131, 67)$.

Graphing More Than One Scatterplot and Trendline on the Same Axes

There are two different ways to go about this.

Method 1. Creating the graph from scratch.

Bring up the data set, which will contain an independent variable and two or more dependent variables. Make sure the independent variable is in the left column. You cannot

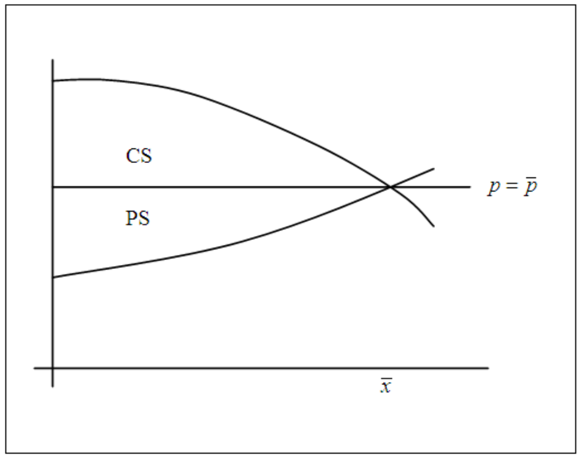

Figure 17.10: Graph for exploring what happens if the established price is higher than market equilibrium.

	A	B	C	D
1	x value	D(x) - S(x)	D(x)	S(x)
2	120	=C2-D2	=-0.0029*A2*A2-0.0139*A2+118.26	=0.0015*A2*A2+0.0806*A2+30.596
3				

Figure 17.11: Setting up Excel to use goal seek to find the point of intersection of the supply and demand curves.

use StatPro to create a graph with more than one dependent variable. Your must use Excel's chart wizard.

1. Highlight the independent and dependent data you wish to graph.
2. Go to "Insert" and click on "Scatter"
3. Select the sub-type you want
4. Add the appropriate information about your graph in the Chart Title and the x- and y-axes.
5. Add the appropriate trend line(s) displaying the equation and the R^2.
6. Repeat step 5 for the each data series (set of y values) on the graph.

Method 2. Adding a scatterplot to an existing graph.

If your chart already exists and you want to add a new series of y-values to the graph, you do not have to start over and make a new chart (although you could, using Method 1 above to get all the y-data on the graph). The first step is to right-click in the white space around the existing graph. Choose "Select data..." from the menu that appears to bring up the dialog box in figure 17.14. For example, suppose we had the function D from the

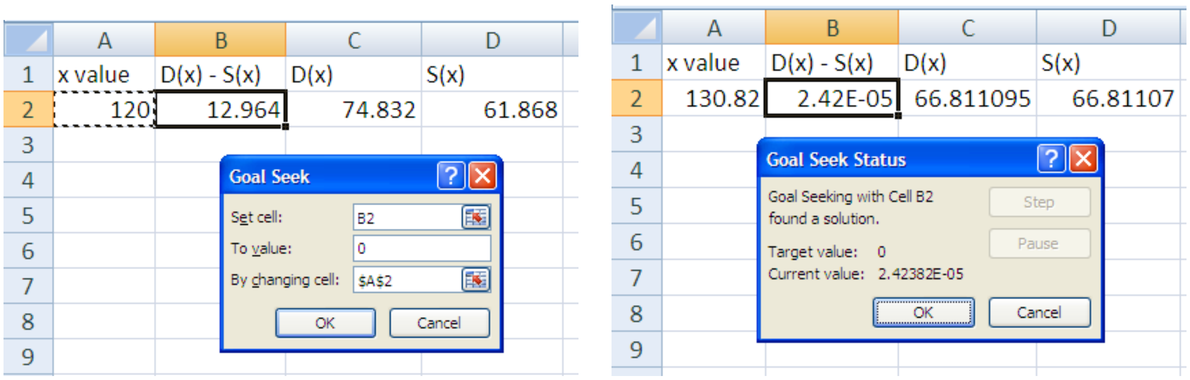

Figure 17.12: Goal seek finds the point where the curves intersect.

example above plotted by itself (see figure 17.13) and we wanted to add a graph of function S to the same axes.

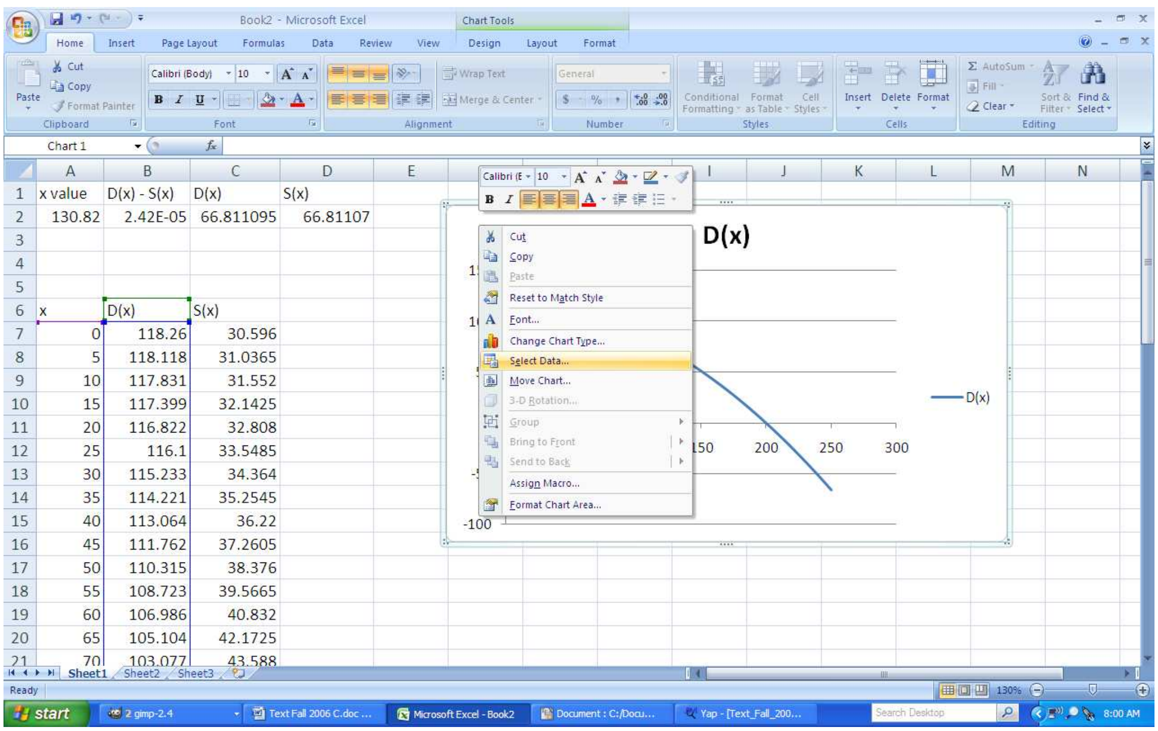

Figure 17.13: Adding a second set of y values to a graph by right-clicking on the graph.

The easiest way to add the new column of data is to simply highlight all of the data (the new and the old) that you now want to appear in the graph. If the columns of data you want graphed are not adjacent to each other, remember to hold down the control key while you select the regions. Alternatively, you can select "Add" to add a new data series, but that takes a little more work.

When you are finished, hit "OK" and the new data should appear on the chart.

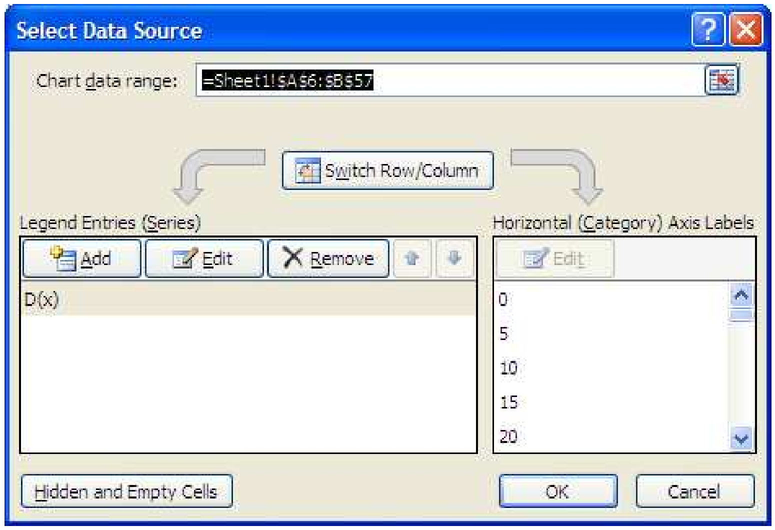

Figure 17.14: Selecting new data for a graph.

Graphing More Than One Equation on the Same Axes

Suppose you know the equations you wish to plot. Perhaps you have graphed two scatterplots and their trendlines as above. You wish, however, to graph just the trendlines by themselves without the scatterplots. Bring up an Excel spreadsheet. You need to determine how many x-values you wish to use. If you have already done scatterplots and trendlines of the data, you can use it as a guide. To illustrate, suppose you want x to range from 1 to 160 and suppose you wish to graph two functions:

$$\begin{aligned} D(x) &= -0.0029x^2 - 0.0139x + 118.26 \\ S(x) &= 0.0015x^2 + 0.0806x + 30.596 \end{aligned}$$

Set up your spreadsheet as you see below. Enter 1-160 under the x column (highlight 1, 2, 3 and drag the fill handle down to A161) and enter the two equations in B2 and C2, respectively. Double click the fill handle on B2 and C2 to generate the remaining values. Next, highlight all the data (the column of x data and the columns of y data) and insert a new scatter plot. Select the sub-type of chart on the left side of the second row (see figure 17.15).

Drawing on a chart

Occasionally, you will want to add details to a chart in order to call attention to certain features or to fill in missing information. The drawing tools can help you do this. To access the drawing tools, go to "View/Toolbars/Drawing". One example of this would be adding in the line which marks the equilibrium level in a graph showing the supply and demand curves. You can do this one of two ways.

Method 1. Draw the line yourself Select the line tool from the drawing menu; it looks exactly like a straight line. Position the mouse where you want the line to start and then

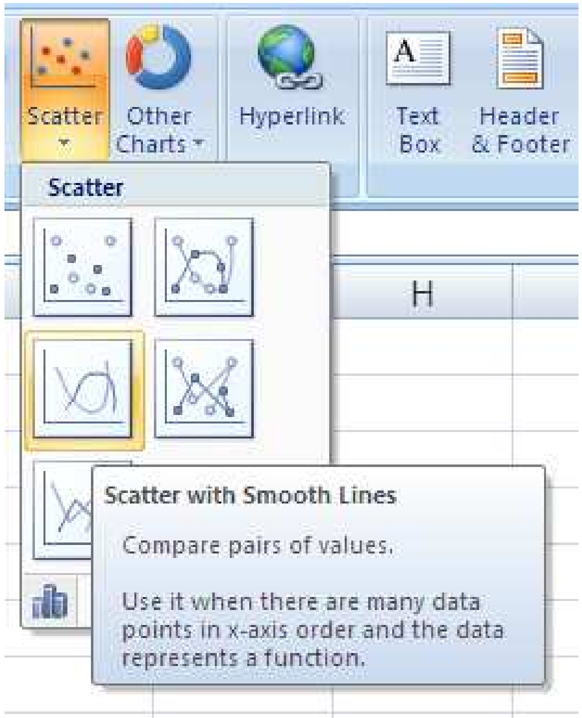

Figure 17.15: Adding a scatterplot without the data points.

LEFT-CLICK and DRAG the other end of the line until it is where you want it. Release the mouse and your line is there. Double-clicking on the line will allow you to change features like its color, thickness, and so forth.

TIP: To force the line to be exactly horizontal, hold down the SHIFT key while dragging and do not release the shift key until after you release the mouse button.

Method 2. Add the line as the graph of a horizontal line You could also add a new column of y-values to the data table. Fill in any y-value you wish in the first cell in the column (such as cell D2 in figure 17.24). Then, in the second cell (D3) enter the formula "=D2" and copy this formula down the column. Add this series to the graph (or create a new graph with all the series included). Now you can adjust the value in cell D2 and the entire line will remain horizontal. Keep adjusting the position of the line until it is exactly where you want it.

17.3 Homework

17.3.1 Mechanics and Techniques Problems

17.1. A standard normal distribution is a normal distribution whose mean is 0 and whose standard deviation is 1. The function that gives rise to the standard normal distribution is:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

1. Verify that the area under the standard normal distribution from $-\infty$ to $+\infty$ is 1. In the Basic Integration tool, use -50 for $-\infty$ and 50 for $+\infty$. Use 3.141593 for π.

2. Verify the Rules of Thumb (also called the "Empirical Rule") for the standard normal distribution (see Definitions and Formulas in Chapter 4B.

17.2. A few years ago, you set up an internet business that initially brought in \$30,000 revenue and \$800 per year thereafter. At start up time, you immediately invested this income stream at 6% interest compounded continuously. You want \$1,000,000 to accrue from this investment in order to retire. After how many years will you be able to retire? Hint: Set $x_0 = 0$ in the Basic Integration Tool, replace the function in C9 with the appropriate function for this problem, and change x_N (which is T in the integral you need) until you hit your goal.

17.3. The management of Fitter Than Thou Health Spa is considering renovating its exercise room and buying new equipment. It has developed two plans. Plan 1 costs \$700,000 to renovate the room, buy the equipment and then install it. It is expected to generate an income stream of \$550,000 per year over the next 5 years. Plan 2 requires less initial outlay at \$250,000 but will generate an income stream of only \$470,000 per year for the next 5 years. If the interest rate is expected to hold at 8% per year for the next 5 years,

1. calculate the present value of the income stream of each plan for the 5 year period; and

2. determine which plan will generate a higher net income after the 5-year period.

17.4. The demand function for a collapsible pull-along sports carrier is

$$p = d(x) = \ln(75 - 0.005x^2)$$

where p is the unit price in hundreds of dollars and x is the quantity demanded per week. The corresponding supply function is

$$p = s(x) = \sqrt{1 + 0.03x}$$

where p is the unit price and x is the number of carriers the supplier is willing to make available at price p.

1. Find the consumers' surplus and the producers' surplus if the unit price is set at the equilibrium price.

2. Graph the consumers' and producers' surplus on the same axes when the unit price is set at the equilibrium price.

17.3.2 Application and Reasoning Problems

17.5. At last year's annual association fair of suppliers of Digiview camcorders, surveys were taken of how many of a new digital model the association members would be willing to supply at various prices. A supply function $S(x)$ was generated by regressing on this data. In a similar fashion, a demand function $D(x)$ was generated from surveys taken in malls across the country as to the price consumers would be willing to pay for this new model. After determining the consumers' surplus and the producers' surplus, the association recommended at that time that the digital model's market price be set at the equilibrium price. Recently, however, the association journal ran an article expressing alarm over the relatively large producers' surplus compared to the consumers' surplus. Without modifying the demand function, the article recommended that suppliers need to rethink how much they would be willing to supply of the model at various prices and that new surveys need to be taken at this year's fair in order to establish a new market price.

1. What do you think the article recommended?

2. Explain the reasoning behind the recommendation.

17.6. Suppose the demand and supply curves for a commodity are known. Discuss the state of the market for this commodity from both the consumers and producers points of view when

1. the established price is set below the equilibrium price $\bar{p}$, and

2. the established price is set above the equilibrium price $\bar{p}$.

17.7. Discuss how we can determine when market equilibrium has been reached by adding together the areas for the consumers' surplus and the producers' surplus for increasing values of x (i.e. the area between the supply and demand curves)? Hint: see figure 17.16.

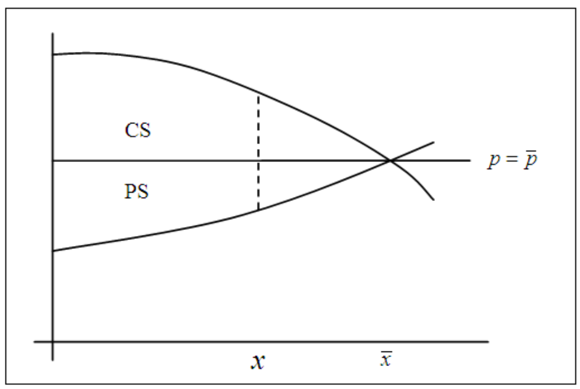

Figure 17.16: When do we reach market equilibrium?

17.3.3 Memo Problem

To: Analysis Staff
From: Project Management Director
Date: May 30, 2008
Re: Pricing Dispute

Ted Bair, one of the managers of Cool Toys for Tots, has requested our help in resolving a pricing conflict within the company over a new digital doll it wants to market. Ted's group has spent valuable resources in gathering data from consumer and producer market surveys in order to help establish a rational price for this doll. Other managers, however, have a gut feeling based on their many years of experience in the business that the selling price should to be $55? per doll. Ted would like to determine the price based on research data, whatever price that may turn out to be. His group could do this for themselves but he thinks it would be better if someone outside the company did the analysis and made a recommendation. So here's what I think we should do:

1. Based on the survey data (see the attachment), find the demand and supply functions.
2. Calculate the consumers' and producers' surplus for the equilibrium point.
3. Determine the consumers' and producers' surplus for the company based on the intended pricing of $55 per doll and the demand at this price.
4. Present graphs of the consumers' and producers' surplus from 2) and 3)
5. Make a recommendation as to how the company should set its price $\bar{p}$ and what the demand $\bar{x}$ would be at this price.

If it's possible to make a compromise or "diplomatic" recommendation one way or the other, I'm sure Ted would appreciate it and pass along more business our way.

Attachments: Data File C17 Pricing.XLS

Appendix A

Excel Cursor Shapes

The pointer on the screen can take any of nine different shapes. The shape of the pointer is a clue to what actions will take place when you click the left mouse button.

Shape of pointer	**Action when you left click**
Normal arrow	Selects the current item
Fat plus sign	Selects the cell (either for entering data/formulas or for other purposes, like copying)
Skinny plus sign	Click and drag to copy the formula(s) or the pattern in the selected cell(s) to other cells on the worksheet
I-beam	Enter text
Short down arrow	Selects entire columns of data
Short right arrow	Selects entire rows of data
Double arrow (one line)	Click and drag to change cell widths (left-right arrows) or heights (up-down arrows)
Double arrow (two lines)	Allows you to split the worksheet into separate areas
Four-headed arrow	Click and drag to move toolbars around